Linear Algebra and its Applications 434 (2011) 424–442

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Relaxed linear spaces and a generalization of the Cayley–Dickson process

Raúl Felipe [a,b,*], Raúl Felipe-Sosa [b]

[a] CIMAT, Callejón Jalisco s/n, Mineral de Valenciana, Guanajuato, Gto., Mexico
[b] ICIMAF, Calle E esquina a 15, No. 309, Vedado, Ciudad de la Habana, Cuba

ARTICLE INFO

Article history:
Received 8 April 2010
Accepted 27 August 2010
Available online 19 September 2010

Submitted by R.A. Brualdi

AMS classification:
11R52
15A03

Keywords:
Relaxed vector spaces
Relaxed linear algebra
Cayley–Dickson process

ABSTRACT

In this paper first we introduce a new generalization of vector spaces and linear nonassociative algebras, and then we apply these new concepts to produce new structures related to the classical real division algebras but with dimensions other than 1, 2, 4 and 8.

1. Introduction

In this brief introduction we comment the three main ideas which are discussed for us in this work. Next, we could do this respecting the order they appear in the article.

Let $(\Lambda, +)$ be an abelian group. The idea of Λ acting as scalars somewhere can be realized by considering abelians groups $(V, +)$ together with homomorphisms ('actions') of abelian groups

$$\Lambda \otimes_{\mathbb{Z}} V \longrightarrow V.$$

Rings $(A, +, .)$ (associativity is not required) with one of these 'actions' are good candidates to be 'algebras' in the general setting chosen for us. In case that $(\Lambda, +)$ is the additive group of a field, we

* Corresponding author at: CIMAT, Callejón Jalisco s/n, Mineral de Valenciana, Guanajuato, Gto., Mexico.
E-mail addresses: raulf@cimat.mx (R. Felipe), rfelipe@icmf.inf.cu, rfs@icmf.inf.cu (R. Felipe-Sosa).

0024-3795/$ - see front matter
doi:10.1016/j.laa.2010.08.040

call $(V, +)$ a relaxed linear space while linear relaxed algebra is reserved for $(A, +, .)$. The unit of the ring A, if any, is called total unit. In case that $(A, +, .)$ is a ring with involution, say $*$, we use the name pre-relaxed $*$-algebras. Then, total units have partial counterparts, the so called partial units, probably one of the three main ideas contained in our work.

Our generalization of vector spaces and algebras is studied for the structures appearing in a modified Cayley–Dickson process, which is the second main idea in the present paper. Also, some 'relaxed' algebraic notions are presented to develop the theory in connection with these 'algebras'. This modification works as follows: take a usual real algebra/pre-relaxed $*$-algebra with an involution $x \longrightarrow x^*$, $\lambda \neq 0$ and $\gamma > 1$ and define on $A \times A$ the product

$$(x, y)(u, v) = (xu - \lambda^2 v^* y, \gamma(vx + yu^*)). \tag{1}$$

The structure of relaxed vector space is then defined by:

• *Case 1:* $\mathbb{R}$ cannot be one to one identified as a subalgebra of A, then as the scalar relaxed multiplication by real numbers we take the usual one multiplication of elements of $A \times A$ by real numbers.

• *Case 2:* $\mathbb{R}$ can be one to one identified as a subalgebra of A. In this case, the scalar relaxed multiplication and the relaxed multiplication between relaxed vectors coincide and itself will be define by means of (1).

Finally, we propose a generalization of the Cayley–Dickson process by considering two real $*$-algebras C, D related by a couple of linear maps $h : D \longrightarrow C$ and $m : C \longrightarrow D$ instead of a single algebra A. The product on $C \times D$ is given by

$$(c_1, d_1)(c_2, d_2) = \left(c_1c_2 - \left(h(d_2)^*\right) h(d_1), d_2m(c_1) + d_1(m(c_2))^*\right).$$

This approach produces algebras of possibly odd dimension, in contrast with the usual Cayley–Dickson process. This is the third main idea of the article.

In summary this paper introduces a generalization of vector spaces, called relaxed linear spaces, together with a corresponding notion of relaxed linear (nonassociative) algebras and various related structures such as involutions and quadratic forms. The main goal of the paper is to generalize the construction of the classical real division algebras, namely the complex numbers, the quaternions, and the octonions. The motivation for introducing these structures is to provide new examples of associative and nonassociative structures which are closely related to real division algebras but have dimensions other than 1, 2, 4 and 8.

2. Relaxed linear spaces and relaxed linear algebras

2.1. Relaxed linear spaces

Before going to the mathematics we would like to express that this paper was inspired in ideas which first appeared in our manuscript [7] and that it is integral part of a broader program in progress.

There are several algebras which can be constructed by the Cayley–Dickson process and that also play a fundamental role in physic. As it is well known, among them are the algebras of quaternions and octonions (see [3] for more details). This process has provided fundamental services for the study of some topics in several branches of the mathematics, for instance, the functional analysis and the algebra (composition algebras and others) also in the algebraic geometry (projective geometry and others) to quote only a few of them.

In our opinion, even today, the subject maintain its permanence and great potential to establish new grounds.

In this section, we show a generalized Cayley–Dickson process which is the key ingredient in this work. But before, we need to adapt some classic definitions to a more general context, which allows us to formulate in an appropriate way the new process. Essentially, we simplify assumptions in well known definitions which alter slightly the nature of the results. In the past, many people have looked at standard axiom systems and wondered what could be deduced if one or several axioms are changed or eliminated. For instance we can cite the consideration of "non-commutative fields" by Hamilton, or the various ways to weaken or alter the associative law with alternative algebras, Lie algebras, and Jordan algebras.

More exactly, we present a Cayley–Dickson process over sets in which we have three operations, the sum of each pair of elements of the set, scalar multiplication between elements belong to the set and scalars living in a fixed field, and finally the product of pairs of elements of the set, without to assume all the axioms of vector space and linear algebra. For this type of sets, in our paper, we have reserved the name of relaxed algebras.

We should add that our process coincides over linear algebras with the familiar Cayley–Dickson process.

Definition 1. A **relaxed vector space** (or **relaxed linear space**) consists of the following:

1. A field $\mathbb{F}$ of the scalars.
2. A set V of objects, called relaxed vectors.
3. A rule (or operation), called relaxed addition, which associates with each pair of relaxed vectors x, y in V a relaxed vector $x + y$ in V, called the sum of x and y, in such a way that the relaxed addition is commutative and associative. There is a unique zero relaxed vector for the addition and moreover each relaxed vector has a unique relaxed opposite vector.
4. A rule (or operation), called the scalar relaxed multiplication, which associates with each scalar c in $\mathbb{F}$ and relaxed vector x in V a relaxed vector cx, called the product of c with x, in such a way that: $c(x + y) = cx + cy$ and $(c_1 + c_2)x = c_1x + c_2x$.

Basically, we have omitted from the usual definition of vector space the axioms: (a) $1x = x$ for all $x \in V$ and (b) $(c_1c_2)x = c_1(c_2x)$. So, obviously all vector space is a relaxed vector space.

In what follows, we could consider only relaxed vector space V for which $2v = 0$ implies that $v = 0$. That is, we assume throughout the remainder of the paper that the “characteristic" of V is not two.

We must observe that $(-1)x$ does not necessarily coincide with the opposite of x for the addition. Hence, one shall be careful at the moment to apply the distributive law. In what following we use x_{op} to denote the opposite of x for the addition.

Example 2. Consider $\mathbb{C}$ as a real vector space and with the help of the usual addition and product by scalars define a new product by real number as follows: let us assume that $\alpha \neq 0$ and $\beta \neq 0$ are two real numbers both do not simultaneously equal to 1. Next, we define

$$\lambda \bullet (a + ib) = \alpha\lambda a + \beta\lambda bi,$$

then it is not hard to see that $\mathbb{C}$ with same addition $+$ and the product $\bullet$ (that is $(\mathbb{C}, +, \bullet)$) is a real relaxed vector space which is not a vector space. In fact, for instance $1 \bullet (a + bi) = \alpha a + \beta bi \neq (a + ib)$.

Example 3. We recall that the elements of the space $l^2(\mathbb{C})$ are sequences of complex numbers $f = \{f_n\}$ such that

$$\sum_{n=1}^{\infty} |f_n|^2 < \infty.$$

We now retain in $l^2(\mathbb{C})$ the usual addition by sequences and introduce a new product by scalar in the following form: as usual let $\{e_n\}$ be the canonical bases of unit vectors $e_1 = (1, 0, 0, \ldots), e_2 = (0, 1, 0, 0, \ldots), \ldots$; then we define

$$c \bullet e_i = ce_{i+1}.$$

It shows that if $f \in l^2(\mathbb{C})$ then $c \bullet f = (0, cf_1, cf_2, \ldots)$. One can see that $(l^2(\mathbb{C}), +, \bullet)$ is a real relaxed vector space and that it is not a vector space. In this example if $c_1 \neq 0$ and $c_2 \neq 0$ then $(c_1c_2)f \neq c_1(c_2f)$.

2.2. Relaxed linear algebra

Definition 4. Let $\mathbb{F}$ be a field. A **linear relaxed algebra** is a relaxed vector space A over $\mathbb{F}$ with an additional operation called relaxed multiplication of relaxed vectors which associates with each pair of relaxed vectors x, y in A a relaxed vector xy in A called the relaxed product of x with y in such a way that: the multiplication is distributive with respect to the addition.

It is very important to note that we do not assume that the relaxed multiplication of relaxed vectors is associative. Also observe that the axiom: $c(xy) = (cx)y = x(cy)$ always included in the usual definition of linear algebra was dropped. Thus all linear algebra is a linear relaxed algebra. We say that e is a **total unit** of a linear relaxed algebra A if this element satisfies the following property, $ae = ea = a$ for any $a \in A$.

Proposition 5. *Let A be a real linear relaxed algebra without unit. Then the usual unitization A_1 of A is a real linear relaxed algebra with a total unit.*

Proof. It is trivial and so it will be omitted. □

Definition 6. A **relaxed involution** over a real linear relaxed algebra A is a mapping $x \to x^*$ of A onto itself satisfying the following involution axioms: (1) $(x^*)^* = x$ for all $x \in A$; (2) $(x+y)^* = x^* + y^{\star}$ for all $x, y \in A$; finally must be assume (3) $(x \cdot y)^* = y^* \cdot x^*$ for all $x, y \in A$. All real linear relaxed algebra which possess a relaxed involution will be called a **pre-relaxed $*$-algebra**.

Let $Sym^+(A)$ be the set contains those $a \in A$ to which $a^* = a$ and define the set $Sym^-(A)$ of all $a \in A$ satisfying that $a^* = a_{op}$. It is clear that $Sym^+(A) \cap Sym^-(A) = \{0\}$.

Definition 7. A real linear relaxed algebra A endowed with a relaxed involution is called a **relaxed $*$-algebra** if $(ra)^* = ra^*$ for each $r \in \mathbb{R}$ and every $a \in A$. We also shall assume that $(x_{op})^* = (x^*)_{op}$.

Lemma 8. *Let A be a relaxed $*$-algebra then for every $r \in \mathbb{R}$, $(ra)^*$ belongs to $Sym^+(A)$ if $a \in Sym^+(A)$ and $(ra)^* \in Sym^-(A)$ if $a \in Sym^-(A)$. Also any element of A can be written in the form $x = x_r + x_i$ where $x_r \in Sym^+(A)$ and $x_i \in Sym^-(A)$.*

Proof. Since $a \in Sym^+(A)$ then $(ra)^* = ra^* = ra$. Let us suppose that $a \in Sym^-(A)$ then $(ra)^* = ra^* = ra_{op} = (ra)_{op}$. To prove the second part of this Lemma note that if we define $x_r = \frac{1}{2}(x + x^*)$ and $x_i = \frac{1}{2}(x + (x^*)_{op})$, then clearly $x_r^* = x_r$, $x_i^* = (x_i)_{op}$. We only prove the last equality. In fact, $x_i^* = \left(\frac{1}{2}(x + (x^*)_{op})\right)^* = \frac{1}{2}(x + (x^*)_{op})^* = \frac{1}{2}(x^* + ((x^*)_{op})^*) = \frac{1}{2}(x^* + x_{op}) = (x_i)_{op}$. □

Definition 9. Let A be a relaxed $*$-algebra. An element e belonging to A is called a **partial unit** of A if by definition

(a) for all $\delta \in \mathbb{R}$ non-zero, we have $\frac{1}{\delta}(\delta e) = e = \delta\left(\frac{1}{\delta}e\right)$,
(b) there are two non-zero real numbers μ and ν, such that,

$$(\mu e)\, a = a = a\,(\mu e) \quad \text{for all} \quad a \in Sym^+(A)$$

$$(\nu e)\, a = a = a(\nu\, e) \quad \text{for any} \quad a \in Sym^-(A).$$

(c) $ea = ae = \frac{1}{\mu}a \in Sym^+(A)$, if $a \in Sym^+(A)$ and $ea = ae = \frac{1}{\nu}a \in Sym^-(A)$, if $a \in Sym^-(A)$.

So far as we know, this definition of partial unit, has never introduced before. We say that a relaxed $*$-algebra is partially unital if it has a partial unit. If e is a partial unit, then the pair (μ, ν) is called the **mark** of the unit.

Lemma 10. *Let e be a partial unit for a relaxed $*$-algebra A. Then $e^* = e$, that is, $e \in Sym^+(A)$.*

Proof. For each $x \in A$ we have

$$ex = e(x_r + x_i) = (x_r + x_i)e = \left(\frac{1}{\mu}x_r + \frac{1}{\nu}x_i\right), \tag{2}$$

so from (2), in particular, we obtain

$$ee^* = e(e_r^* + e_i^*) = \left(\frac{1}{\mu}e_r^* + \frac{1}{\nu}e_i^*\right) = \left(\frac{1}{\mu}e_r + \frac{1}{\nu}(e_i)_{op}\right), \tag{3}$$

and from this last equality follows that

$$ee^* = (ee^*)^* = \left(\frac{1}{\mu}e_r + \frac{1}{\nu}e_i\right), \tag{4}$$

now, (3) and (4) imply that $2e_i = 0$, which simply means that $e_i = 0$. $\square$

Corollary 11. *Let A be a relaxed $*$-algebra. Assume that e_1 and e_2 are two partial units of A with marks (μ_1, ν_1) and (μ_2, ν_2), respectively. If $\mu_1 = \mu_2$ then $e_1 = e_2$.*

Proof. In fact, of the definition of partial unit is deduced that $\frac{1}{\mu_2}e_1 = e_1e_2 = \frac{1}{\mu_1}e_2$. Thus, $\frac{1}{\mu_2}e_1 = \frac{1}{\mu_1}e_2$. Hence, if $\mu_1 = \mu_2$ then one infers that $e_1 = e_2$. $\square$

Now, let us assume that e is a partial unit of a relaxed $*$-algebra A whose mark is (μ, ν), then we may easily see that $e_c = \mu e$ is also a partial unit of A for which the mark is $\left(1, \frac{\nu}{\mu}\right)$.

Definition 12. A partial unit is said to be canonical if its mark is of the way $(1, \kappa)$.

Therefore, all partial unit has associated a canonical partial unit. Reciprocally all canonical partial unit of A generates infinite partial units, in fact, let e_c a canonical partial unit and let $(1, \kappa)$ be its mark, then for all $\mu \neq 0 \in \mathbb{R}$, $e = \mu e_c$ is a partial unit with mark $\left(\frac{1}{\mu}, \frac{\kappa}{\mu}\right)$. This process will be called the expansion of the canonical partial unit.

Lemma 13. *In a relaxed $*$-algebra with partial units exists only one canonical partial unit. Hence, except expansions of the canonical partial unit, it exits only one partial unit.*

Proof. It follows from Corollary 11. $\square$

Proposition 14. *Assuming that e_c is the canonical partial unit of a relaxed $*$-algebra A with mark (1,1), we then get that e_c is a total unit of A.*

Proof. Note that $e_cx = xe_c$ for any $x \in Sym^+$ and for each $x \in Sym^-$. Now the proposition can be proved if we remember that all element $x \in A$ is written in the form $x = x_r + x_i$ where $x_r = \frac{1}{2}(x + x^*) \in Sym^+$ and $x_i = \frac{1}{2}(x + (x^*)_{op}) \in Sym^-$. $\square$

Definition 15. Let A be a partially unital relaxed $*$-algebra and let e_c be its canonical partial unit. We say that $x \in A$ is regular if there exists an unique $y \in A$, such that, $xy = yx = e_c$. Then y is called the inverse of x.

Lemma 16. *Let $x \in Sym^{+}(A)$ be a regular relaxed vector. Let y be its inverse, then $y \in Sym^{+}(A)$.*

Proof. Let us suppose that $x \in Sym^{+}(A)$ is regular and let y be its inverse, clearly in this case we have $(xy)^{*} = y^{*}x^{*} = y^{*}x = e_c$. In the same way we obtain that $(yx)^{*} = x^{*}y^{*} = xy^{*} = e_c$. Hence, by the uniqueness of the inverse $y = y^{*}$, that is, $y \in Sym^{+}(A)$. □

We can to generalize the Definition 9 in the following form:

Definition 17. Let A be a relaxed $*$-algebra. An element e belonging to A is called a **partial unit** of A of length n ($n \geqslant 2$) if

(a) for all $\delta \in \mathbb{R}$ non-zero, we have $\frac{1}{\delta}(\delta e) = e = \delta\left(\frac{1}{\delta}e\right)$,
(b) there is a non-zero real numbers μ such that,
$$(\mu e)\, a = a = a\, (\mu e) \quad \textit{for all} \quad a \in Sym^{+},$$
(c) $Sym^{-} = (Sym^{-})_1 \oplus \cdots \oplus (Sym^{-})_{n-1}$ and there are $n-1$ non-zero real numbers $\nu_1, \ldots, \nu_{n-1}$ for which
$$(\nu_k e)\, a = a = a(\nu_k\, e) \quad \textit{for any} \quad a \in (Sym^{-})_k,$$
where $k = 1, \ldots, n-1$.
(d) $ea = ae = \frac{1}{\mu}a \in Sym^{+}$, if $a \in Sym^{+}$ and for each k, $ea = ae = \frac{1}{\nu_k}a \in (Sym^{-})_k$, if $a \in (Sym^{-})_k$.

Of course the partial units introduced in the Definition 9 are units by part of length 2 with respect to the present definition.

We observe that many properties of the partial units of length 2 already proved are also hold for partial units of arbitrary length, however we do not enunciate or prove neither one of these here.

2.3. The relaxed Cayley–Dickson process

After of these preview considerations we pass to introduce our Cayley–Dickson process which allows us to introduce a particular relaxed $*$-algebra:

Let A be a real algebra and let an involution $x \longrightarrow x^{*}$ be defined in A. Suppose that $\lambda \neq 0$ and $\gamma > 1$ are fixed. Then over the set of all pairs (s, t) where $t, s \in A$, may be defined a multiplication of Cayley–Dickson type in the form
$$(x, y)(u, v) = (xu - \lambda^{2}v^{*}y, \gamma(vx + yu^{*})) \tag{5}$$
for $x, y, u, v \in A$.

Theorem 18. *Let A be a real $*$-algebra. Then the multiplication (5) endows $A \times A$ with a structure of pre-relaxed $*$-algebra.*

Proof. In fact, first we endow $A \times A$ with the obvious addition, that is, by components. Clearly, it is a relaxed addition. To introduce the relaxed multiplication by scalars we need to analyze two cases.

Case 1: $\mathbb{R}$ cannot be one to one identified as a subalgebra of A, then as the scalar relaxed multiplication by real numbers we take the usual one multiplication of elements of $A \times A$ by real numbers. This makes $A \times A$ in a relaxed vector space.

Case 2: $\mathbb{R}$ can be one to one identified as a subalgebra of A. In this case, the scalar relaxed multiplication and the relaxed multiplication between relaxed vectors coincide and itself will be define by means of (5).

It is straightforward to show that (5) is distributive with respect to the relaxed addition.

Next, we introduce as possible relaxed involution the customary application on $A \times A$. Thus, we put as usual $(c, d)^{*} = (c^{*}, -d)$. Clearly it is a relaxed involution. □

We would like to observe that if A has a unit e and $a^* = a$ for every $a \in A$, then the relaxed vector $(e, 0)$ is a partial unit with mark $\left(1, \frac{1}{\gamma}\right)$ for $A \times A$. Thus, it is also its canonical partial unit.

Definition 19. The linear relaxed $*$-algebra $A \times A$ obtained with help of theorem 18 will be called a generalized (or relaxed) Cayley–Dickson extension.

Remark 20. Really the Theorem 18 represent a first step in the recursive construction of pre-relaxed $*$-algebras. That is, if in place of to take a real algebra with involution, we consider a pre-relaxed $*$-algebra A, then the product (5) on $A \times A$ gives us again a pre-relaxed $*$-algebra. In particular we can use this product over two copies of any relaxed Cayley–Dickson extension to obtain a new relaxed Cayley–Dickson extension. This process may be performed in a recursive way an infinite number of times.

It is impossible in this short note to prove some general properties of the generalized Cayley–Dickson extension, or to try to obtain a classification theorem for linear relaxed $*$-algebra, because between others reasons, first we shall need to know which are the properties that shall have the norm or some relaxed notion of norm, as also the composition law or some relaxed concept of composition law on these new structures. So, first we must study some particular generalized (or relaxed) Cayley–Dickson extension which can help us to answer these questions.

Therefore, instead of this, we shall next restrict the discussion to whose cases for which $A = \mathbb{R}$, $A = \mathbb{C}$ or a relaxed $*$-algebra obtained from these. As it was already indicated the principal motive for this is that we are interested in to understand and to show promptly the additional algebraic notions which, in our opinion, must be taken into account in a future to study of classification for linear relaxed $*$-algebra.

3. Relaxed Cayley–Dickson algebras in low dimensions

In this section, we will use the process introduced in the previous section when the case (2) of the proof of the Theorem 18 appears. Really, all relaxed algebra introduced in this section is a relaxed vector space which is not a usual vector space.

Moreover, we present examples of relaxed algebras obtained from $\mathbb{R}$ in a recursive form and for which

$$\|x\|\,\|y\| \leqslant \|xy\|,$$

where $\|.\|$ is the relaxed norm defined by the relaxed involution. It is clear that in this case the Hurwitz property $\|x\|\,\|y\| = \|xy\|$ has been replaced by one more general. It is important to note that all the relaxed algebras found are nonassociative.

3.1. Dimension 2

Now, let us suppose that $A = \mathbb{R}$. Keeping in $A = \mathbb{R}$ the identity as its original involution, then if we apply over $\mathbb{R}$ the generalized Cayley–Dickson precess according to the Theorem 18, it induces over $\mathbb{R} \times \mathbb{R}$ a structure of relaxed $*$-algebra which is not an usual algebra and that we denote by $\mathbb{C}_{(\lambda,\gamma)}$. Here, the relaxed involution coincide with the habitual involution.

Obviously $\{(1,0),(0,1)\}$ is a relaxed basis of $\mathbb{C}_{(\lambda,\gamma)}$ as relaxed vector space, let us denote these relaxed vectors as $1_{(\lambda,\gamma)}$ and $i_{(\lambda,\gamma)}$, respectively. From now on $\mathbb{C}_{(\lambda,\gamma)}$ will be called the complex (λ, γ)-plane. If we apply the product law to the elements of this basis we obtain the following table:

$\cdot$	$1_{(\lambda,\gamma)}$	$i_{(\lambda,\gamma)}$
$1_{(\lambda,\gamma)}$	$1_{(\lambda,\gamma)}$	$\gamma i_{(\lambda,\gamma)}$
$i_{(\lambda,\gamma)}$	$\gamma i_{(\lambda,\gamma)}$	$-\lambda^2 1_{(\lambda,\gamma)}$

, (6)

it clear also that if $z \in \mathbb{C}_{(\lambda,\gamma)}$ then $z = a1_{(\lambda,\gamma)} + bi_{(\lambda,\gamma)}$ and on the other hand $z_1z_2 = (a_1 1_{(\lambda,\gamma)} + b_1 i_{(\lambda,\gamma)})(a_2 1_{(\lambda,\gamma)} + b_2 i_{(\lambda,\gamma)}) = (a_1a_2 - \lambda^2 b_1b_2)1_{(\lambda,\gamma)} + \gamma(b_1a_2 + a_1b_2)i_{(\lambda,\gamma)}$. Using this formula we can see that $\mathbb{C}_{(\lambda,\gamma)}$ is commutative.

Lemma 21. *Suppose that $\gamma > 1$, then $\mathbb{C}_{(\lambda,\gamma)}$ is not associative with respect to the product defined by (6).*

Proof. Since $\gamma > 1$, then in $\mathbb{C}_{(\lambda,\gamma)}$ we have $(i_{(\lambda,\gamma)}1_{(\lambda,\gamma)})1_{(\lambda,\gamma)} = \gamma^2 i_{(\lambda,\gamma)}$ while $i_{(\lambda,\gamma)}(1_{(\lambda,\gamma)}1_{(\lambda,\gamma)}) = \gamma i_{(\lambda,\gamma)}$, thus the linear relaxed algebra $\mathbb{C}_{(\lambda,\gamma)}$ is not associative. $\square$

Lemma 22. *The relaxed vector $1_{(\lambda,\gamma)}$ is the canonical partial unit of length 2 of $\mathbb{C}_{(\lambda,\gamma)}$ and its mark is $\left(1, \frac{1}{\gamma}\right)$.*

Proof. It is evident. $\square$

The example is interesting in that it gives a nontrivial generalized Cayley–Dickson extension of dimension two obtained from $\mathbb{R}$ which is not associative.

Definition 23. We define the relaxed norm in $\mathbb{C}_{(\lambda,\gamma)}$ if $\lambda \neq 0$ as

$$\|z\|_{(\lambda,\gamma)} = \sqrt{(a^2 + \lambda^2 b^2)}.$$

The relaxed norm is related with two facts which we have grouped in a Lemma.

Lemma 24. *The equality $zz^* = \|z\|_{(\lambda,\gamma)}1_{(\lambda,\gamma)}$ holds for all $z \in \mathbb{C}_{(\lambda,\gamma)}$. In this case, the term relaxed norm puts in evidence the circumstantial fact that for $\gamma > 1$ and all $\epsilon \in \mathbb{R}$, the expression $\|\epsilon z\|_{(\lambda,\gamma)} \neq |\epsilon|\|z\|_{(\lambda,\gamma)}$ hold for almost all ϵ and z.*

Proof. Observe that

$$\begin{aligned} zz^* &= (a1_{(\lambda,\gamma)} + bi_{(\lambda,\gamma)})(a1_{(\lambda,\gamma)} - bi_{(\lambda,\gamma)}), \\ &= a^2 1_{(\lambda,\gamma)} - ab\gamma i_{(\lambda,\gamma)} + ab\gamma i_{(\lambda,\gamma)} + \lambda^2 b^2 1_{(\lambda,\gamma)}, \\ &= (a^2 + \lambda^2 b^2)1_{(\lambda,\gamma)}. \end{aligned}$$

To see that $\|\epsilon z\|_{(\lambda,\gamma)} \neq |\epsilon|\|z\|_{(\lambda,\gamma)}$ we first note that

$$\|\epsilon z\|_{(\lambda,\gamma)} = |\epsilon|\sqrt{(a^2 + \lambda^2\gamma^2 b^2)},$$

and at the same time we see that

$$|\epsilon|\|z\|_{(\lambda,\gamma)} = |\epsilon|\sqrt{(a^2 + \lambda^2 b^2)},$$

the Lemma is proved. $\square$

Observe that $\|z\|_{(\lambda,\gamma)} = \|\widehat{z}\|_{(1,1)}$, where $\widehat{z} = a + i\lambda b$. We remember that we have assumed that $\lambda \neq 0$.

Definition 25. A **relaxed norm** over a relaxed vector space X is a function $\|.\| : X \times X \longrightarrow \mathbb{R}_+$, such that, $\|x\| \geqslant 0$ for all $x \in X$, and $\|x\| = 0$ if and only if $x = 0$. Moreover, it is required that $\|x + y\| \leqslant \|x\| + \|y\|$.

A relaxed vector space X with a relaxed norm is called a **relaxed normed space**. In the same way, a linear relaxed algebra equipped with a relaxed norm will be called a **normed linear relaxed algebra**.

Among others differences this definition differs of the usual one in that the property $\|\epsilon x\| = |\epsilon|\|x\|$ was dropped, However, this detail will be relevant below. Clearly all norm is a relaxed norm. Thus $\mathbb{C}_{(\lambda,\gamma)}$

is a normed linear relaxed algebra. It is trivial to show that the unitization A_1 of a real normed linear relaxed algebra A without unit is also a real normed linear relaxed algebra.

Lemma 26. *All non-null relaxed vector $z = x + i_{(\lambda,\gamma)}y$ of $\mathbb{C}_{(\lambda,\gamma)}$ is regular.*

Proof. In fact, it is possible to see that $z^{-1} = \left(\frac{x}{\|z\|^2_{(\lambda,\gamma)}} - i_{(\lambda,\gamma)}\frac{y}{\|z\|^2_{(\lambda,\gamma)}}\right)$. □

Notice that one can write this inverse in a more compact form as $z^{-1} = \frac{z^*}{\|z\|^2_{\mathbb{C}_{(\lambda,\gamma)}}}$. We now give the following definition.

Definition 27. Let H be a relaxed vector space over $\mathbb{F}$. A **relaxed inner product** is a function $(.,.) : H \times H \longrightarrow \mathbb{F}$, such that,

(1) $(x,x) \geqslant 0$, for each $x \in H$ and $((x,x) = 0) \Longleftrightarrow (x = 0)$,
(2) $\overline{(x,y)} = (y,x)$ for every $x, y \in H$,
(3) $(x, y + z) = (x,y) + (x,z)$ for all x, y and z belong to H,

where by $\mathbb{F}$ we mean $\mathbb{R}$ or $\mathbb{C}$. We say that the relaxed vector space H is a space with a relaxed inner product if over it is defined a relaxed inner product.

By means of an adequate relaxed inner product in $\mathbb{C}_{(\lambda,\gamma)}$ we can recover $\|.\|_{(\lambda,\gamma)}$. In fact

Lemma 28. *Define $(z_1,z_2)_{(\lambda,\gamma)} = z_1\overline{z_2}$, then $(z_1,z_2)_{(\lambda,\gamma)}$ is a relaxed inner product and moreover $\|z\|_{(\lambda,\gamma)} = (z,z)_{(\lambda,\gamma)}$.*

Proof. It is trivial. □

Now, we have

Theorem 29. *For any $z_1, z_2 \in \mathbb{C}_{(\lambda,\gamma)}$*

$$\|z_1\|_{(\lambda,\gamma)}\|z_2\|_{(\lambda,\gamma)} = \|z_1z_2\|_{(\lambda,1)}, \tag{7}$$

where we insist that the required product z_1z_2 between z_1 and z_2 is computed in $\mathbb{C}_{(\lambda,1)}$.

Proof. In fact

$$\begin{aligned}
\|z_1\|^2_{(\lambda,\gamma)}\|z_2\|^2_{(\lambda,\gamma)} &= (a_1^2 + \lambda^2 b_1^2)(a_2^2 + \lambda^2 b_2^2),\\
&= (a_1a_2)^2 + \lambda^2(a_1b_2)^2 + \lambda^2(b_1a_2)^2 + \lambda^4(b_1b_2)^2,\\
&= (a_1a_2)^2 - 2\lambda^2(a_1a_2)(b_1b_2) + \lambda^4(b_1b_2)^2 + \lambda^2(a_1b_2)^2 + \lambda^2(b_1a_2)^2\\
&\quad + 2\lambda^2(a_1a_2)(b_1b_2),\\
&= (a_1a_2 - \lambda^2 b_1b_2)^2 + \lambda^2(a_1b_2 + b_1a_2)^2,\\
&= \|z_1z_2\|^2_{(\lambda,1)},
\end{aligned}$$

thus the theorem is proved. □

It is clear that $\mathbb{C}_{(\lambda,1)} \cong \mathbb{C}$. However, in general $\mathbb{C}_{(\lambda,\gamma)} \ncong \mathbb{C}$, which is evident if $\gamma > 1$. Now (7) would yield to a much more interesting fact.

Corollary 30. *For $\gamma > 1$ and for all $z_1, z_2 \in \mathbb{C}_{(\lambda,\gamma)}$ we have*

$$\|z_1\|_{(\lambda,\gamma)}\|z_2\|_{(\lambda,\gamma)} \leqslant \|z_1 z_2\|_{(\lambda,\gamma)}. \tag{8}$$

Other simple form of to prove this result, in the case $\lambda = 1$, is the following: notice that the value of the relaxed norm for any relaxed vector z of $\mathbb{C}_{(1,\gamma)}$ coincides with the value of its norm but now as element of $\mathbb{C}$ (that is, $\mathbb{C}$ with the usual multiplication), therefore $\|z_1\|_{(1,\gamma)}\|z_2\|_{(1,\gamma)} = \|z_1\|_{(1,1)}\|z_2\|_{(1,1)} = \|z_1 z_2\|_{(1,1)}$. In the last term of this norm-preserving formula, the product is been taking in $\mathbb{C}$. On other hand, it is evident that $\|z_1 z_2\|_{(1,1)} \leqslant \|z_1 z_2\|_{(1,\gamma)}$ when $\gamma > 1$, where in the expression $\|z_1 z_2\|_{(1,\gamma)}$ the product is given in the $\mathbb{C}_{(1,\gamma)}$ context. This argument will be used below again very soon.

To our knowledge, inequalities of this type appeared for first time in a work by Arens [2] in real normed algebras (see also [4]), however concrete examples of these algebras were not given for him. Arens proved that if A is a real normed algebra satisfying $k\|y\|\,\|x\| \leqslant \|yx\|$ for some positive constant k and all x, y, then A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

It is for this reason that we introduce the following definition.

Definition 31. We will called **Arens relaxed algebra** to any normed linear relaxed algebra for which its relaxed norm satisfies an inequality of the type (8) which is called the **reversed multiplicative condition**.

Certainly, the condition (8) of the relaxed norm in an Arens relaxed algebra precludes the existence of non-zero zero divisors.

Lemma 32. *For all relaxed vectors z_1 and z_2 of $\mathbb{C}_{(\lambda,\gamma)}$ we have (reversed Cauchy—Schwarz inequality in $\mathbb{C}_{(\lambda,\gamma)}$)*

$$\|z_1\|_{(\lambda,\gamma)}\|z_2\|_{(\lambda,\gamma)} \leqslant \|(z_1, z_2)_{(\lambda,\gamma)}\|_{(\lambda,\gamma)}, \tag{9}$$

where as before $(z_1, z_2)_{(\lambda,\gamma)} = z_1\overline{z_2}$.

Proof. Obviously (9) follows from (8). $\square$

3.2. Dimension 4

Now we construct the relaxed Cayley–Dickson algebra $\mathbb{H}_{(\lambda,\gamma)}$ over $\mathbb{R}$ as the relaxed algebra structure on $\mathbb{C}_{(\lambda,\gamma)} \times \mathbb{C}_{(\lambda,\gamma)}$ given by the formulae:

Let $z = (z_1, z_2)$ and $w = (w_1, w_2)$ in $\mathbb{C}_{(\lambda,\gamma)} \times \mathbb{C}_{(\lambda,\gamma)}$ then

$$zw = \left(z_1 w_1 - \lambda^2 w_2^* z_2, \gamma\left(w_2 z_1 + z_2 w_1^*\right)\right),$$

where $z^* = (z_1^*, -z_2)$.

We denote $1_c = ((1,0),(0,0))$, $i = ((0,1),(0,0))$, $j = ((0,0),(1,0))$ and $k = ((0,0),(0,1))$, then it is possible to write each element of $\mathbb{H}_{(\lambda,\gamma)}$ in the form $q = a1_c + bi + cj + dk$ and it is easy to see that $q^* = a1_c - bi - cj - dk$.

We can construct the following multiplication table

$\cdot$	1_c	i	j	k
1_c	1_c	$i\gamma$	γj	$\gamma^2 k$
i	$i\gamma$	$-\lambda^2 1_c$	$\gamma^2 k$	$-\gamma\lambda^2 j$
j	γj	$-\gamma^2 k$	$-\lambda^2 1_c$	$\gamma\lambda^2 i$
k	$\gamma^2 k$	$\gamma\lambda^2 j$	$-\lambda^2\gamma i$	$-\lambda^4 1_c$

(10)

from which follows that we can introduce in $\mathbb{H}_{(\lambda,\gamma)}$ a relaxed norm by mean of the relaxed involution in the form $(z_1, z_2)(z_1^*, -z_2) = (\|z_1\|^2_{(\lambda,\gamma)} + \lambda^2\|z_2\|^2_{(\lambda,\gamma)})1_c$, hence we define

$$\|(z_1, z_2)\|^2_{\mathbb{H}_{(\lambda,\gamma)}} = \|z_1\|^2_{(\lambda,\gamma)} + \lambda^2\|z_2\|^2_{(\lambda,\gamma)}, \tag{11}$$

which coincide with the norm of the space $\mathbb{H}$ of quaternions, namely when $\lambda = 1$ and $\gamma = 1$ (more even it is also hold if $\lambda = 1$ and $\gamma > 1$).

Similarly of (10) one can check that the (λ, γ)-Hamilton product is determined in the form

$$\begin{aligned} q_1q_2 &= (a_1 1_c + a_2 i + a_3 j + a_4 k)(b_1 1_c + b_2 i + b_3 j + b_4 k) \\ &= (a_1b_1 - a_2b_2\lambda^2 - a_3b_3\lambda^2 - a_4b_4\lambda^4)1_c + \gamma(a_1b_2 + a_2b_1 - a_4b_3\lambda^2 + a_3b_4\lambda^2)i \\ &\quad + \gamma(a_1b_3 + a_3b_1 - a_2b_4\lambda^2 + a_4b_2\lambda^2)j + \gamma^2(a_1b_4 + a_4b_1 + a_2b_3 - a_3b_2)k. \end{aligned} \tag{12}$$

If $\lambda = 1$, this product differs of the multiplication in $\mathbb{H}$ by the presence of γ before the coefficients of i, j and k.

It is clear that $\mathbb{H}_{(\lambda,\gamma)}$ is a relaxed $*$-algebra. Now, the main distinctive feature of $\mathbb{H}_{(1,\gamma)}$ is

Theorem 33. *If $\gamma > 1$ then $\mathbb{H}_{(1,\gamma)}$ is an Arens relaxed algebra and its canonical partial unit e_c is 1_c which has length 4.*

Proof. We only verify that the relaxed norm satisfies an inequality of the type (8). Before anything observe that, as already it was commented, if $q = (z_1, z_2) \in \mathbb{H}_{(\lambda,\gamma)}$

$$\begin{aligned} \|(z_1, z_2)\|^2_{\mathbb{H}_{(1,\gamma)}} &= \|z_1\|^2_{(1,\gamma)} + \|z_2\|^2_{(1,\gamma)}, \\ &= \|z_1\|^2_{\mathbb{C}} + \|z_2\|^2_{\mathbb{C}}, \\ &= \|(z_1, z_2)\|^2_{\mathbb{H}}, \end{aligned} \tag{13}$$

hence, for every $q \in \mathbb{H}_{(1,\gamma)}$ we see that $\|q\|_{(1,\gamma)} = \|q\|_{\mathbb{H}}$.

On other hand, we have two facts. First, by a well known theorem from Hurwitz follows that $\|q_1\|_{\mathbb{H}_{(1,\gamma)}}\|q_2\|_{\mathbb{H}_{(1,\gamma)}} = \|q_1\|_{\mathbb{H}}\|q_2\|_{\mathbb{H}} = \|q_1q_2\|_{\mathbb{H}}$. Second, the presence of $\gamma > 1$ in (12) implies that $\|q_1q_2\|_{\mathbb{H}} \leqslant \|q_1q_2\|_{\mathbb{H}_{(1,\gamma)}}$. Clearly here, in the first term of this inequality, the product is in $\mathbb{H}$ while that in the second term, the multiplication is that of $\mathbb{H}_{(1,\gamma)}$. The Theorem is proved. $\square$

3.3. Dimension 8

Next, we can define the relaxed $*$-algebra of (λ, γ)-octonions $\mathbb{O}_{(\lambda,\gamma)}$ from $\mathbb{H}_{(\lambda,\gamma)}$ through our relaxed Cayley–Dickson process. It will be the real relaxed Cayley–Dickson algebra obtained on $\mathbb{H}_{(\lambda,\gamma)} \times \mathbb{H}_{(\lambda,\gamma)}$ with the product

$$zw = \left(z_1w_1 - \lambda^2 w_2^* z_2, \gamma\left(w_2z_1 + z_2w_1^*\right)\right), \tag{14}$$

where $z^* = (z_1^*, -z_2)$ is the relaxed involution. Now $z = (z_1, z_2)$ and $w = (w_1, w_2)$ belong to $\mathbb{H}_{(\lambda,\gamma)} \times \mathbb{H}_{(\lambda,\gamma)}$. Then we have

Theorem 34. *The relaxed algebra $\mathbb{O}_{(1,\gamma)}$ is an Arens relaxed algebra for $\gamma > 1$ and $1_c = (((1, 0), (0, 0)), ((0, 0), (0, 0)))$ is a partial unit of length 8.*

In the proof of this theorem is not necessary to make cumbersome calculations with the elements of $\mathbb{O}_{(\lambda,\gamma)}$. In fact

Proof. Let $z = (z_1, z_2) \in \mathbb{O}_{(\lambda,\gamma)}$ arbitrary. Observe that from (14) follows that the second component of zz^* is zero. More exactly

$$zz^* = \|z\|^2_{\mathbb{O}_{(\lambda,\gamma)}} 1_c = \left(\|z_1\|^2_{\mathbb{H}_{(\lambda,\gamma)}} + \lambda^2 \|z_2\|^2_{\mathbb{H}_{(\lambda,\gamma)}}\right) 1_c \tag{15}$$

relation of which disappeared the parameter γ. It is easy to show that $\|.\|_{\mathbb{O}_{(\lambda,\gamma)}}$ is a relaxed norm. Now, if $\lambda = 1$ then $\|.\|_{\mathbb{O}_{(1,\gamma)}} = \|.\|_{\mathbb{O}}$. Finally note that in this case (that is, when $\lambda = 1$) the presence of γ in the second component of (14) implies that $\|zw\|_{\mathbb{O}} \leqslant \|zw\|_{\mathbb{O}_{(1,\gamma)}}$, where the products in each term correspond to the products in the respective spaces such as it is indicated for the subindex. In fact, let $e_1, e_2, \ldots, e_8$ be the canonical basis in $\mathbb{R}^8$. Define $i_1 = e_2, i_2 = e_3, \ldots, i_7 = e_8$, clearly $1_c = e_1$. Then from (12) follows that zw in $\mathbb{O}_{(1,\gamma)}$ is a linear combination of the relaxed vectors $1_c, i_1, i_2, \ldots, i_7$ whose coefficients coincide with the coefficients of these vectors for the product zw in $\mathbb{O}$ multiplied by a power of γ. This argument justifies the inequality.

Thus, from already mentioned Hurwitz Theorem we have

$$\|z\|_{\mathbb{O}_{(1,\gamma)}} \|w\|_{\mathbb{O}_{(1,\gamma)}} = \|z\|_{\mathbb{O}} \|w\|_{\mathbb{O}} = \|zw\|_{\mathbb{O}} \leqslant \|zw\|_{\mathbb{O}_{(1,\gamma)}}. \tag{16}$$

We recall that the product zw in the two final terms of (12) indicates the product in the respective space noted for the subindex. $\square$

Let $\{e_i\}_{i=1}^8$ be the canonical basis of $\mathbb{O}_{(\lambda,\gamma)}$, for instance

$$e_6 = (((0,0),(0,0)),((0,1),(0,0))),$$

then the multiplication table associated to the elements of this basis is

$\cdot$	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_1	e_1	γe_2	γe_3	$\gamma^2 e_4$	γe_5	$\gamma^2 e_6$	$\gamma^2 e_7$	$\gamma^3 e_8$
e_2	γe_2	$-\lambda^2 e_1$	$\gamma^2 e_4$	$-\gamma\lambda^2 e_3$	$\gamma^2 e_6$	$-\gamma\lambda^2 e_5$	$-\gamma^3 e_8$	$\gamma^2\lambda^2 e_7$
e_3	γe_3	$-\gamma^2 e_4$	$-\lambda^2 e_1$	$\gamma\lambda^2 e_2$	$\gamma^2 e_7$	$\gamma^3 e_8$	$-\gamma\lambda^2 e_5$	$-\gamma^2\lambda^2 e_6$
e_4	$\gamma^2 e_4$	$\gamma\lambda^2 e_3$	$-\lambda^2\gamma e_2$	$-\lambda^4 e_1$	$\gamma^3 e_8$	$-\gamma^2\lambda^2 e_7$	$\gamma^2\lambda^2 e_6$	$-\gamma\lambda^4 e_5$
e_5	γe_5	$-\gamma^2 e_6$	$-\gamma^2 e_7$	$-\gamma^3 e_8$	$-\lambda^2 e_1$	$\gamma\lambda^2 e_2$	$\gamma\lambda^2 e_3$	$\gamma^2\lambda^2 e_4$
e_6	$\gamma^2 e_6$	$\gamma\lambda^2 e_5$	$-\gamma^3 e_8$	$\gamma^2\lambda^2 e_7$	$-\gamma\lambda^2 e_2$	$-\lambda^4 e_1$	$-\gamma^2\lambda^2 e_4$	$\gamma\lambda^4 e_3$
e_7	$\gamma^2 e_7$	$-\gamma^3 e_8$	$\gamma\lambda^2 e_5$	$-\gamma^2\lambda^2 e_6$	$-\gamma\lambda^2 e_3$	$\gamma^2\lambda^2 e_4$	$-\lambda^4 e_1$	$-\gamma\lambda^4 e_2$
e_8	$\gamma^3 e_8$	$-\gamma^2\lambda^2 e_7$	$\gamma^2\lambda^2 e_6$	$\gamma\lambda^4 e_5$	$-\gamma^2\lambda^2 e_4$	$-\gamma\lambda^4 e_3$	$\gamma\lambda^4 e_2$	$-\lambda^6 e_1$

4. Relaxed composition algebras

It is well known that the theory of division algebras can be presented in terms of composition algebras. In this section we extend this approach to relaxed algebras.

To begin this section, we define the notion of quadratic form on a relaxed vector space V over a field $\mathbb{F}$. We adopt in a full way the classic one.

Definition 35. A quadratic form on a relaxed vector space is a mapping $N : V \longrightarrow \mathbb{F}$ with the properties:

(i) $N(\lambda x) = \lambda^2 N(x)$ $(\lambda \in \mathbb{F}, x \in V)$.
(ii) The mapping $\langle .,. \rangle : V \times V \longrightarrow \mathbb{F}$ defined by

$$\langle x, y\rangle = N(x+y) - N(x) - N(y),$$

is bilinear, i.e., it is linear in each of x and y separately.

As a consequence of the observations of the previous section, with regard to the space $\mathbb{C}_{(\lambda,\gamma)}$, note that on it is defined a **quadratic form** $N_{(\lambda,\gamma)} : \mathbb{C}_{(\lambda,\gamma)} \longrightarrow \mathbb{R}$ which coincide with $\|.\|^2_{(\lambda,\gamma)}$. We exhibit in explicit form this quadratic form.

Lemma 36. *If $z = a1_{(\lambda,\gamma)} + bi_{(\lambda,\gamma)} \in \mathbb{C}_{(\lambda,\gamma)}$, then $N_{(\lambda,\gamma)}(z) = (a^2 + \lambda^2 b^2)$ is a quadratic form and its bilinear form associated is $\langle z, w\rangle = 2(ac + \lambda^2 bd)$ where $w = c1_{(\lambda,\gamma)} + di_{(\lambda,\gamma)}$. Also, we have*

$$N_{(\lambda,\gamma)}(z_1)N_{(\lambda,\gamma)}(z_2) \leqslant N_{(\lambda,\gamma)}(z_1 z_2). \tag{17}$$

Proof. It is easy to show that (17) follows from inequality (8). □

In view of (17) and to make itself yet clearer we propose the following definition.

Definition 37. A linear relaxed algebra A which is not necessarily associative with respect to the relaxed multiplication between relaxed vectors is called **relaxed composition algebra** if there exists a nondegenerate quadratic form N on A such that

$$N(z_1)N(z_2) \leqslant N(z_1 z_2), \tag{18}$$

for any $z_1, z_2 \in A$. Observe that we do not demand that A has an identity element.

The previous definition weakens significantly the notion of composition algebra (see [13]). On other hand every composition algebra is a relaxed composition algebra.

Definition 38. Let us assume that A is a relaxed composition algebra, then an element $e \in A$ will be a called a **Hurwitz unit** if $N(e) = 1$ and $e^2 = e$. A relaxed composition algebra which admits a Hurwitz unit e will be called **unital relaxed composition algebra**.

Easy inspection shows that the unit of a composition algebra is a Hurwitz unit in the sense of relaxed composition algebras.

Lemma 39. *It is clear that $\mathbb{C}_{(\lambda,\gamma)}$ is a unital relaxed composition algebra.*

It is convenient to observe that if C and D are two real composition algebras with quadratic form N_c and N_d, respectively, then for any α, β reals $\alpha N_c + \beta N_d$ is a quadratic form on $C \times D$.

Definition 40. We say that a quadratic form N on a real relaxed vector space is positive if $N(x) \geqslant 0$ for all $x \in A$ and $N(x) = 0$ if and only if $x = 0$.

Proposition 41. *Let D be a relaxed composition algebra with respect to a positive quadratic form $N(.)$, then D cannot contain zero divisors.*

Proof. Because for any $x, y \in D$

$$0 \leqslant N(x)N(y) \leqslant N(xy). \quad \square$$

Then the following general statement holds.

Theorem 42. *Let A be a real associative and unital composition algebra for which its quadratic form $N(.)$ is positive and $\lambda, \gamma \in \mathbb{R}$, such that $\lambda \neq 0$ and $\gamma > 1$. Define on $A \times A$ a Cayley–Dickson type product (5) with respect to the standard involution on A. Then by the Theorem 18, $A \times A$ equipped with this product is a pre-relaxed $*$-algebra. In addition we may choose for $A \times A$ a quadratic form which is defined as $N_{\lambda,\gamma}((x, y)) = N(x) + \lambda^2 N(y)$. Combining both things we find that $A \times A$ is a unital relaxed composition algebra. A Hurwitz unit in this claimed relaxed composition algebra is $(e, 0)$, where e is the unit of A.*

This relaxed composition algebra is called the relaxed composition (λ, γ)-algebra associated to the associative composition algebra A.

Proof. First of all, note that the quadratic form on $A \times A$ is nondegenerate, more even it is positive. To show that $N_{\lambda,\gamma}$ admits a composition law which satisfies (18), we compute as follows:

$$\begin{aligned} N_{\lambda,\gamma}((x,y)(u,v)) &= N(xu - \lambda^2 v^* y) + \lambda^2\gamma^2 N(vx + yu^*) \\ &= N(xu) + \lambda^4 N(v^* y) - \lambda^2 \langle x\,u, v^*\,y\rangle \\ &\quad + \lambda^2\gamma^2 (N(vx) + N\,(yu^*) + \langle v\,x, y\,u^*\rangle). \end{aligned} \tag{19}$$

Now, let $*$ be the standard involution on A, it is well known that for $x, y, z \in A$

$$\langle x\,y, z\rangle = \langle y, x^*\,z\rangle, \tag{20}$$
$$\langle x\,y, z\rangle = \langle x, z\,y^*\rangle, \tag{21}$$

thus, as A is associative, from (20) and (21) we deduce that $\langle x\,u, v^*\,y\rangle = \langle v\,x, y\,u^*\rangle$.

On other hand, taking into account that $\gamma > 1$ we obtain

$$\begin{aligned} N_{\lambda,\gamma}((x,y))N_{\lambda,\gamma}((u,v)) &= (N(x) + \lambda^2 N(y))(N(u) + \lambda^2 N(v)), \\ &= N(xu) + \lambda^2 N(vx) + \lambda^2 N\,(yu^*) + \lambda^4 N(v^* y), \\ &= (N(xu) + \lambda^4 N(v^* y) - \lambda^2\langle x\,u, v^*\,y\rangle) \\ &\quad + \lambda^2 (N(vx) + N\,(yu^*) + \langle v\,x, y\,u^*\rangle), \\ &\leqslant (N(xu) + \lambda^4 N(v^* y) - \lambda^2\langle x\,u, v^*\,y\rangle) \\ &\quad + \lambda^2\gamma^2 (N(vx) + N\,(yu^*) + \langle v\,x, y\,u^*\rangle), \\ &= N(xu - \lambda^2 v^* y) + \lambda^2\gamma^2 N(vx + yu^*), \end{aligned} \tag{22}$$

where we have used the fact that $N(.)$ is positive. Also we have had into account that for every element x, $N(x) = N(x^*)$ and moreover the trivial relation $N(ab) = N(a)N(b) = N(b)N(a) = N(ba)$.

Hence, from (19) and (22) it is follows that $N_{\lambda,\gamma}((x,y))N_{\lambda,\gamma}((u,v)) \leqslant N_{\lambda,\gamma}((x,y)(u,v))$. The fact that $(e, 0)$ is a Hurwitz unit is evident. Thus the Theorem is proved. □

Example 43. The latter Theorem allows to us to define the set $\mathbb{H}^N_{(\lambda,\gamma)}$ of (λ,γ)—quaternions, these quaternions are a 4-dimensional real linear relaxed algebra with basis $1 = (1, 0)$, $i = (i, 0)$, $j = (0, 1)$, and $k = (0, i)$. To describe the product we can give a table

$\cdot$	1	i	j	k
1	1	i	γj	γk
i	i	-1	γk	$-\gamma j$
j	γj	$-\gamma k$	$-\lambda^2 1$	$\lambda^2 i$
k	γk	γj	$-\lambda^2 i$	$-\lambda^2 1$

, (23)

evidently this real linear relaxed algebra is not commutative neither associative. (even more, $\mathbb{H}^N_{(\lambda,\gamma)}$ is a real relaxed normed space). Clearly, $\mathbb{H}^N_{(\lambda,\gamma)}$ is the relaxed composition (λ,γ)-algebra associated to $\mathbb{C}$.

Notice also that in general $\mathbb{H}^N_{(\lambda,\gamma)} \neq \mathbb{H}_{(\lambda,\gamma)}$. Next, we include an explanation of the relation between the table in equation (10) and the table in equation (23). As it already was observed these tables are different and the reason for this is that the power of the parameters in each entry is in general bigger in the table (10) that in the table (23). It is caused to the fact of that in the first case the relaxed Cayley–Dickson process was applied to $\mathbb{C}_{(\lambda,\gamma)} \times \mathbb{C}_{(\lambda,\gamma)}$ while in the second case the process was applied to $\mathbb{C} \times \mathbb{C}$.

The Theorem 42 can be also used to construct the relaxed composition (λ,γ)-algebra $\mathbb{O}^N_{(\lambda,\gamma)}$ associated to $\mathbb{H}$.

5. Relaxed Cayley–Dickson algebras of unusual dimension

5.1. Generalized Cayley-Dickson extension

As we will see in this section the framework of the Cayley–Dickson process can change if instead of to take only one algebra A we draw on of two algebras with involution, C (by Cayley) and D (by Dickson) and we substitute $A \times A$ by $C \times D$. Thus, the main objective of this section is to show that the Cayley–Dickson process remains holds even in the general case for which the formed algebra should be $C \times D$. *It is clear that this situation to open the possibility of that the new algebra with underlying vector space $C \times D$ can possess odd dimension*. One should notice that certainly it is a new setting in the literature about the subject, since up to now all the theory developed concentrates in the case in which $C = D$. We must remit to the reader to some of the papers in the references, where it can be found different applications and interesting generalizations *in other directions* of the Cayley–Dickson process.

Theorem 44. *Let C and D be two real $*$-algebras. Assume that $h : D \longrightarrow C, m : C \longrightarrow D$ are both linear applications, such that,*

$$h\left(d^*\right) = (h(d))^*, \tag{24}$$

for any $d \in D$. Then, the multiplication

$$(c_1, d_1)(c_2, d_2) = \left(c_1c_2 - h\left(d_2^*\right)h(d_1), d_2m(c_1) + d_1m\left(c_2^*\right)\right), \tag{25}$$

endows $C \times D$ with a structure of real $*$-algebra. The case $C = D$ is not excluded.

Really it is routine to check that this Theorem holds, but since it represents the key tool in this section we will give its proof next.

Proof. Note that if $C = D$ and h and m are both the identity map, we recover the classic Cayley–Dickson process, but it is possible to have the new situation: $C = D$ and, at the same time, almost one of the maps h or m to be different from the identity map (*we believe that this possibility makes it case attractive for studying*). Only from the linearity of h and m follows that $C \times D$ is an algebra. In fact, first we endows $C \times D$ with the obvious addition and multiplication by scalars, that is by components, to convert this space in a vector space. On one hand, if $z_1 = (c_1, d_1), z_2 = (c_2, d_2)$ and $z_3 = (c_3, d_3)$ we have

$$\begin{aligned}(z_1 + z_2)z_3 &= (c_1 + c_2, d_1 + d_2)(c_3, d_3)\\ &= \left((c_1 + c_2)c_3 - h\left(d_3^*\right)h(d_1 + d_2), d_3m(c_1 + c_2) + (d_1 + d_2)m\left(c_3^*\right)\right)\\ &= \left((c_1 + c_2)c_3 - h\left(d_3^*\right)(h(d_1) + h(d_2)), d_3(m(c_1) + m(c_2)) + (d_1 + d_2)m\left(c_3^*\right)\right)\\ &= \left(c_1c_3 - h\left(d_3^*\right)h(d_1), d_3m(c_1) + d_1m\left(c_3^*\right)\right)\\ &\quad + \left(c_2c_3 - h\left(d_3^*\right)h(d_2), d_3m(c_2) + d_2m\left(c_3^*\right)\right)\\ &= z_1z_3 + z_2z_3.\end{aligned}$$

On the other hand, in the same form one can prove that $z_1(z_2 + z_3)$. Next, we verify that $(\alpha\beta)(z_1z_2) = (\alpha z_1)(\beta z_2)$.

$$\begin{aligned}(\alpha\beta)(z_1z_2) &= (\alpha\beta)\left(c_1c_2 - h\left(d_2^*\right)h(d_1), d_2m(c_1) + d_1m\left(c_2^*\right)\right),\\ &= \left(\alpha\beta c_1c_2 - h\left(\beta d_2^*\right)h(\alpha d_1), \beta d_2m(\alpha c_1) + \alpha d_1m\left(\beta c_2^*\right)\right),\\ &= (\alpha c_1, \alpha d_1)(\beta c_2, \beta d_2),\\ &= (\alpha z_1)(\beta z_2).\end{aligned}$$

Next, we introduce as possible involution the customary application, but now on $C \times D$. Thus, we put as usual $(c, d)^* = (c^*, -d)$. Turn on that it is again an involution. In fact, it is obvious that $((c, d)^*)^* = (c, d)$. Now

$$
\begin{aligned}
(z_1z_2)^* &= ((c_1,d_1)(c_2,d_2))^*,\\
&= \left(c_1c_2 - h\left(d_2^*\right)h(d_1), d_2m(c_1) + d_1m\left(c_2^*\right)\right)^*,\\
&= \left(c_2^*c_1^* - (h(d_1))^*\left(h\left(d_2^*\right)\right)^*, -\left(d_2m(c_1) + d_1m\left(c_2^*\right)\right)\right),\\
&= \left(c_2^*c_1^* - h\left(d_1^*\right)h(d_2), -\left(d_2m(c_1) + d_1m\left(c_2^*\right)\right)\right),\\
&= \left(c_2^*, -d_2\right)\left(c_1^*, -d_1\right) = (c_2,d_2)^*(c_1,d_1)^* = (z_2)^*(z_1)^*. \qquad \square
\end{aligned}
$$

Definition 45. The algebra $C \times D$ obtained with help of Theorem 44 will be called a generalized Cayley–Dickson extension.

Proposition 46. *Let C and D be two real and unital $*$-algebras. Let us assume h and m as in the previous theorem. Suppose that e_C and e_D are identities in C and D, respectively, such that $m(e_C) = e_D$ and $h(e_D) = e_C$. Then $(e_C, 0)$ is an identity of the generalized Cayley–Dickson extension $C \times D$. On the other hand, if we put $i = (0, e_D)$ then*

$$i^2 = -(e_C, 0).$$

Proof. In fact,

$$(c,d)(e_C,0) = \left(ce_C, dm\left(e_C^*\right)\right) = (ce_C, dm(e_C)) = (c,d),$$

also

$$(e_C,0)(c,d) = (e_Cc, dm(e_C)) = (c,d),$$

on the other hand the result $i^2 = -(e_C, 0)$ is evident. $\square$

Definition 47. We say that a generalized Cayley–Dickson extension is **classic** if it was obtained under the conditions assumed in Proposition 46.

A carefully inspect of the proof of the Theorem 44 tell us that the Condition (24) in it can be removed if we define the product over $C \times D$ in the way

$$(c_1,d_1)(c_2,d_2) = \left(c_1c_2 - (h(d_2))^*h(d_1), d_2m(c_1) + d_1m\left(c_2^*\right)\right). \tag{26}$$

It is easy to show that this product also equips $C \times D$ with a structure of real $*$-algebra. Even so one more generalization of (26) is convenient, in fact, we can define for our intentions over $C \times D$ the following Cayley–Dickson product which to make similar service that (44) and (26)

$$(c_1,d_1)(c_2,d_2) = \left(c_1c_2 - (h(d_2))^*h(d_1), d_2m(c_1) + d_1(m(c_2))^*\right). \tag{27}$$

5.2. Dimension 3

Next we consider the case for which $C = \mathbb{C}$ and $D = \mathbb{R}$. Obviously, $\mathbb{C} \times \mathbb{R}$ has dimension equal to 3. This real vector space is generated by the vectors $t_1 = ((0,0),1)$, $t_2 = ((1,0),0)$ and $t_3 = ((0,1),0)$. For this case, we can choose the maps h and m of the theorem 44 as $h(\lambda) = (\lambda, 0)$, accordingly this is the immersion of $\mathbb{R}$ in $\mathbb{C}$, and at the same time $m(z) = Re(z)$, in others words, m takes the real part of z. It is immediate to see that h and m are linear. Of course, we should take the identity as the real involution in $\mathbb{R}$, then $(h(\lambda))^* = (\lambda, 0) = h(\lambda^*) = h(\lambda)$ for any $\lambda \in \mathbb{R}$. Thus, we have that the space $\mathbb{C} \times \mathbb{R}$ is a 3-dimensional generalized Cayley–Dickson extension which will be denoted by $\mathbb{D}_u$.

It is easy to prove that the table which describe the multiplication of the elements of the basis is:

$\cdot$	t_1	t_2	t_3
t_1	$-t_2$	t_1	t_0
t_2	t_1	t_2	t_3
t_3	t_0	t_3	$-t_2$

, (28)

Table for $h(\lambda) = (\lambda, 0)$ and $m(z) = Re(z)$.

where $t_0 = ((0, 0), 0)$ is the null vector of $\mathbb{C} \times \mathbb{R}$. Note that t_2 is the unit of $\mathbb{D}_u$, on the other hand

$$t_1^2 = t_3^2 = -t_2 \tag{29}$$

and

$$t_1 \cdot t_3 = t_3 \cdot t_1 = t_0 \tag{30}$$

it implies that t_1 and t_3 are zero divisors of $\mathbb{D}_u$. The analysis of (29) and (30) suggests us (the proof is trivial) the presentation of the generalized Cayley–Dickson extension $\mathbb{D}_u$ in the following way: let $\mathbb{D}_u = \{d = a + bi\widetilde{+}cj | a, b, c \in \mathbb{R}\}$ where

$$ij = ji = 0, i^2 = j^2 = -1, \tag{31}$$

here, we must observe that the symbol $\widetilde{+}$ in d means that $a + bi\widetilde{+}cj$ should be identified with the pair $((a, b), c)$ and never with (a, b, c). From this particular point of view the construction of $\mathbb{D}_u$, have an apparent more familiar and turn on more effective to the computations. For instance, one can multiply by components as usual, thus

$$(a_1 + b_1 i\widetilde{+}c_1 j)(a_2 + b_2 i\widetilde{+}c_2 j) = (a_1 a_2 - b_1 b_2 - c_1 c_2) + (a_1 b_2 + a_2 b_1)i\widetilde{+}(a_1 c_2 + a_2 c_1)j,$$

the operations as vector space in $\mathbb{D}_u$ are:

$$(a_1 + b_1 i\widetilde{+}c_1 j) \pm (a_2 + b_2 i\widetilde{+}c_2 j) = (a_1 \pm a_2) + (b_1 \pm b_2)i\widetilde{+}(c_1 \pm c_2)j,$$

and

$$\alpha(a + bi\widetilde{+}cj) = (\alpha a + \alpha bi\widetilde{+}\alpha cj).$$

Let us define $d^* = (a - bi\widetilde{-}cj)$ and then we observe that it coincides with the conjugate of $((a, b), c)$ in the Cayley–Dickson process for $\mathbb{C} \times \mathbb{R}$, therefore it is an involution in this alternative presentation. Now, it is clear that $\mathbb{D}_u$ is **nicely normed** and the norm on this is defined as usual $||d||^2 = dd^*$ having multiplicative inverses $d^{-1} = \frac{d^*}{||d||^2}$.

The algebra $\mathbb{D}_u$ is not associative but it is commutative. Moreover it is classic. In fact, it is well known that an associative algebra has multiplicative inverses if and only if it is a division algebra (we adopt the terminology of [3]). However, our algebra has zero divisors, thus $\mathbb{D}_u$ is not a division algebra. That this algebra is commutative follows from the table (28).

We shall indicate that apparently the elements of $\mathbb{D}_u$ were already known by Hamilton (see [8]) who introduced this numbers in a completely axiomatic way (maybe in 1843). He defined the multiplication of two elements, if $ij = ji$ as

$$\begin{aligned}(a_1 + b_1 i + c_1 j)(a_2 + b_2 i + c_2 j) = {} & (a_1 a_2 - b_1 b_2 - c_1 c_2) + (a_1 b_2 + a_2 b_1)i + (a_1 c_2 + a_2 c_1)j \\ & + (b_1 c_2 + c_1 b_2)ij,\end{aligned}$$

or when $ij = -ji$ in the form

$$\begin{aligned}(a_1 + b_1 i + c_1 j)(a_2 + b_2 i + c_2 j) = {} & (a_1 a_2 - b_1 b_2 - c_1 c_2) + (a_1 b_2 + a_2 b_1)i + (a_1 c_2 + a_2 c_1)j \\ & + (b_1 c_2 - c_1 b_2)ij,\end{aligned}$$

in both cases Hamilton considered the fourth term simply superfluous and he suggested to take $ij = \zeta + \eta i + \theta j$ where ζ, η and θ are fixed numbers. However, we have motives to think that he had in mind that perhaps $ij = 0$ could be suitable. It is not clear what it was his motivation for this proposal. *We know that Hamilton was not in conditions of to use a generalized Cayley–Dickson process*. Hamilton called to this numbers **triplets**. He however associated $a + bi + cj$ with (a, b, c).

5.3. Dimension 6

In this subsection we introduce a generalized Cayley–Dickson extension over $\mathbb{C} \times \mathbb{H}$. For this we use the Theorem 44. If (z_1, q_1) and (z_2, q_2) belong $\mathbb{C} \times \mathbb{H}$, we introduce the relaxed product

$$(z_1, q_1) \cdot (z_2, q_2) = (z_1 z_2 - h(q_2^*)h(q_1), q_2 m(z_1) + q_1 m(z_2^*)),$$

where $m(z) = (z, (0, 0, 0, 0))$ for any $z \in \mathbb{C}$ and $h(q) = h((q^1, q^2, q^3, q^4)) = (q^1, q^2)$ for all $q \in \mathbb{H}$. Let $\{e_m\}_{m=1}^6$ be the natural basis of $\mathbb{C} \times \mathbb{H}$, that is,

$$\begin{aligned} e_1 &= ((1, 0), (0, 0, 0, 0)), \\ e_2 &= ((0, 1), (0, 0, 0, 0)), \\ e_3 &= ((0, 0), (1, 0, 0, 0)), \\ e_4 &= ((0, 0), (0, 1, 0, 0)), \\ e_5 &= ((0, 0), (0, 0, 1, 0)), \\ e_6 &= ((0, 0), (0, 0, 0, 1)). \end{aligned}$$

If we now apply the product between the elements of this basis we arrive to the following table

$\cdot$	t_1	t_2	t_3	t_4	t_5	t_6
t_1	t_1	t_2	t_3	t_4	t_5	t_6
t_2	t_2	$-t_1$	t_4	$-t_3$	$-t_6$	t_5
t_3	t_3	$-t_4$	$-t_1$	t_2	t_0	t_0
t_4	t_4	t_3	$-t_2$	$-t_1$	t_0	t_0
t_5	t_5	$-t_6$	t_0	t_0	t_0	t_0
t_6	t_6	$-t_4$	t_0	t_0	t_0	t_0

, (32)

where $t_0 = ((0, 0), (0, 0, 0, 0))$. Note that t_5 and t_6 are zero self-divisors.

6. Conclusions

In this paper, we have tried to change the appearance to the classic Cayley–Dickson process. In this case, the modifications and generalizations of itself made here have allowed us to find new variants for well known concepts and also to expand the context in which the same applies.

We hope that the approach to algebraic structures through relaxed vector spaces has reserved a promising development.

Acknowledgment

We would like to express our deep gratitude to Daniel Shapiro for very helpful comments and we thank Murray Bremner for electronic contact.

The authors thank the anonymous referee for his/her careful reading and remarks which helped to improve the manuscript.

The first author was supported in part under CONACYT Grant 106923 and both authors were supported in part under the Cuba National Research Project "Theory and efficient algorithms for the solution of problems in operator algebra and geometric modeling".

References

[1] B.N. Allison, J.R. Faulkner, A Cayley–Dickson process for a class of structurable algebras, Trans. Amer. Math. Soc. 283 (1) (1984) 185–210.
[2] R. Arens, Linear topological division algebras, Bull. Amer. Math. Soc. 53 (1947) 623–630.
[3] J. Baez, The octonions, Bull. Amer. Math. Soc. (N. S.) 39 (2) (2001) 145–205.
[4] V.A. Belfi, R.S. Doran, Norm and spectral characterizations in Banach algebras, Enseign. Math. 26 (1980) 103–130.
[5] R.B. Brown, On generalized Cayley–Dickson algebras, Pacific J. Math. 20 (3) (1967) 415–422.
[6] J.A. Cuenca-Mira, On division algebras satisfying Moufang identities, Comm. Algebra 30 (11) (2002) 5199–5206.

[7] R. Felipe, R. Felipe-Sosa, A new version of the Cayley–Dickson process and hyper-composition algebras, unpublished manuscript, July 2008.
[8] W.R. Hamilton, The collected mathematical papers, Royal Irish Academy, vol. 3, 1931, pp. 143.
[9] C. Jiménez-Gestal, J.M. Pérez-Izquierdo, Ternary derivations of generalized Cayley–Dickson algebras, Comm. Algebra 31 (10) (2003) 5071–5094.
[10] N.C. Leung, Riemannian geometry over different normed division algebras, J. Differential Geom. 61 (2002) 289–333.
[11] G. Moreno, The zero divisors of the Cayley–Dickson algebras over the real numbers, 8 October 1997, arXiv:q-alg/9710013v1.
[12] R.D. Schafer, On algebras formed by the Cayley–Dickson process, Amer. J. Math. 76 (1954) 435–446.
[13] T.A. Springer, F.D. Veldkamp, Octonion, Jordan Algebras and Exceptional Groups, Springer-Verlag, 2000.
[14] Y. Yoshii, Cayley polynomials, 23 January 2006, arXiv:math/0601570v1 (math.RA).